Penn State Astrostatistics MCMC tutorial

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A Bayesian change point model

Consider the following hierarchical changepoint model for the number of occurrences Y_i of some event during time interval i with change point k.

$$Y_i|k,\theta,\lambda \sim \text{Poisson}(\theta) \text{ for } i = 1,\dots,k$$
$$Y_i|k,\theta,\lambda \sim \text{Poisson}(\lambda) \text{ for } i = k+1,\dots,n$$

Assume the following prior distributions:

$$\theta \sim \text{Gamma}(0.5, b_1 = 100) \qquad (\text{pdf}=g_1(\theta))$$
$$\lambda \sim \text{Gamma}(0.5, b_2 = 100) \qquad (\text{pdf}=g_2(\lambda))$$
$$k \sim \text{Uniform}(2,\dots,n-1) \qquad (\text{pmf }=u(k))$$

k, θ, λ are conditionally independent and b_1, b_2 are independent.
Assume the Gamma density parameterization Gamma$(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

Inference for this model is therefore based on the 3-dimensional **posterior** distribution $f(k,\theta,\lambda|\mathbf{Y})$ where $\mathbf{Y}$=$(Y_1,\dots,Y_n)$. The posterior distribution is obtained *up to a constant* (that is, the normalizing constant is unknown) by taking the product of all the conditional distributions. Thus we have

$$\begin{aligned} f(k,\theta,\lambda|\mathbf{Y}) &\propto \prod_{i=1}^{k} f_1(Y_i|\theta,\lambda,k) \prod_{i=k+1}^{n} f_2(Y_i|\theta,\lambda,k) \\ &\times g_1(\theta) g_2(\lambda) u(k) \\ &= \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \\ &\times \frac{1}{\Gamma(0.5) b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \times \frac{1}{\Gamma(0.5) b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\ &\times \frac{1}{n-2} 1(k \in \{2,3,\dots,n-1\}) \end{aligned}$$

where $1(k \in \{2,3,\dots,n-2\})$ is an indicator function, meaning it is 1 if $k \in \{2,3,\dots,n-2\}$ and 0 otherwise.

Note that it is easy to add more layers to the model and the priors if that is of interest, for instance b_1, b_2 could themselves be treated as random variables with prior ("hyperprior") distributions.